Scientific Collaboration Networks: The role of Heterogeneity and Congestion*

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Abstract

We propose a dynamic model to analyze the formation of scientific collaboration networks. In this model, individuals continuously take decisions concerning the continuation of existing collaboration links and the formation of new ones with other researchers through a link formation game. Once the network has been constituted, ideas arrive from outside to every node at a constant rate and agents (can) require the collaboration of one of the previously selected coauthors to publish them. Agents are heterogeneous –they have different levels of productivity, and they have a limited processing capability –so congestion can arise if a researcher receives a sufficiently high amount of collaboration requests. As a consequence, the decisions about the link formation trade off the rewards (or costs) from collaborating with more (or less) productive agents against the costs (or rewards) derived from more (or less) congested co-authors.

Focusing on the role of heterogeneity among agents' productivity and congestion problems derived from their limited processing capability we show how self-interested researchers can organize themselves forming the kind of network topologies observed in reality.

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1 Introduction

Social networks underlie many economic and social activities up to the point that outcomes cannot be understood without taking into account the features of the specific network structure. Examples and references are numerous[1]. But one of the frameworks in which the key role of a social network is more evident is scientific production. In academics the association with a group of able colleagues to exchange information is a strong advantage in order to discover errors and discern the correct ways to solve a problem. This unquestionable significance of networks in understanding scientific activity is one of the reasons that explain the extensive empirical work on this field[2]. But there is another reason: it is relatively easy to represent and then obtain network statistics form this collaboration patterns through a direct consequence of them: coauthorship networks. In such a network a link between two nodes (researchers) exists whenever there is some coauthored paper among the two.

Whatever it is the reason, the number of empirical studies of coauthorship networks is large. Newman (2004), Newman (2001a) and Newman (2001b) analyze the defining statistics of coauthorship networks in Biology, Physics and Mathematics. Laband and Tollison (2000) focus on the importance of informal collaboration relationships in the comparison between networks in Economics and Biology. Hudson (1996) looks for the reasons of the increase in the number of coauthors per paper in Economics. But the empirical work that most clearly shows these patterns of collaboration is Goyal, Van der Leij and Moraga (06) (GVM from now on). This work is an excellent basis for having a detailed image of the features of actual coauthorship networks[3].

In spite of the great variety of empirical studies, there is a lack of foundational theoretical models to analyze how the decisions of individuals contribute to the formation of scientific collaboration networks. To the best of our knowledge, chapter 4 in Van der Leij (06) is the only attempt to compensate this deficiency. Here, we propose a different model that chases the same aim.

1.1 Characteristics of coauthorship networks

Before introducing the model, let us describe some of the key features of scientific collaboration networks. These are a type of social network. As such, they have some important properties.

One of the most surprising ones, is the small average distance (measured by the shortest path length) between pairs of nodes. This stylized fact is captured in the famous "six degrees of separation" of John Gaure's play[4]. Scientific collaboration networks are not an exception to this fact as GVM shows. The average distance in the Economics coauthorship network they analyzed was 9.47

[1] Calvó-Armengol and Jackson (2004) on learning about job openings through contacts or Kranton and Minehart (2001) on buyer-seller networks are only two examples

[2] Albert and Barabási (2002) offers a survey of empirical studies on networks

[3] Altough this empirical work refers to the world of Economics, when analyzing the characteristics of co-authorship networks in other fields we will show that most of the features are common

[4] Stanley Milgram (1967) pioneered the study of path length through a clever experiment where people had to send a letter to another person who was not directly known to them. The diameters of a variety of networks have been measured varying from purely social networks, to co-authorship networks, to parts of the internet and world wide web. See Albert and Barabási (2002) for an illuminating account.

with a total population of 33,027 nodes. This stylized fact extends to other fields. Newman (2004) shows that the average distances are 4.6 in Biology, 5.9 in Physics and 7.6 in Mathematics.

Another interesting feature of social networks is that the degree-distribution of the nodes tends to exhibit "fat tails". In particular, GVM found that the 20% of most-linked authors account for about 60% of all the links. This evidence shows that the distribution of links in the population of economists is very unequal. Newman (2004) shows that such a feature can also be extended to coauthorship networks in the fields he studied (Biology, Physics and Mathematics). In each case the distribution is fat tailed, with a small fraction of scientists having a very large number of collaborators.

Focusing on the best-connected researchers both empirical papers can go one step ahead. GVM is able to show that these individuals collaborated extensively and most of their coauthors did not collaborate with each other. Moreover, they also observe that these individuals are essential in maintaining the connectivity of the network. On the other hand, Newman (2004) found that, in the networks studied there, most of the connections (64%) of an individual's shortest path to other researchers pass through the best-connected of their collaborators, and most of the remainders pass through the next-best connected.

These results leads to GVM to conclude that: "the world of Economics is spanned by inter-linked stars" (in essence, an inter-linked star is a network in which some nodes connected among them accumulate a lot of links with other nodes who are not connected among themselves). Despite there is no such a strong conclusion referred to coauthorship networks in other fields, the similarity in the results showed above suggests a similar pattern in Biology, Physics and Mathematics. Moreover, GVM analyzes the evolution over the last thirty years and it is able to conclude that such a structure is stable over time.

1.2 Preview of the model and results

In this model we show that the effects of two simple driving forces can explain the formation of scientific collaboration networks with an interlinked star topology. These two forces are caused by the heterogeneity among researchers and their limited processing capability. To be specific, we propose a dynamic model in which individuals periodically make decisions concerning the continuation of existing collaboration links and the formation of new ones with other researchers through a link formation game. Once the network has been constituted, ideas arrive from outside of the network to every node at a constant rate and agents (can) require the collaboration of one of the previously selected coauthors to publish them. As commented above, agents are heterogeneous –they have different levels of productivity, and they have a limited processing capability –so congestion can arise if a researcher receives a sufficiently high amount of collaboration requests. In consequence, the decisions about the link formation trade off the rewards (or costs) from collaborating with more (or less) productive agents against the costs (or rewards) derived from more (or less) congested coauthors.

We assume that the process reaches a Steady State and we characterize it. Then, we investigate which kind of network topologies can be sustained in Steady State. We first show several results that narrow the set of potential equilibrium networks. In particular, these results show the tendency to accumulate links towards the best players (high productivity) until the point in which they get saturated. Moreover, we show that this can happen with almost-homogeneous agents. Therefore, arbitrarily small differences among agents' productivity can generate big differences in the Steady State agents' utility level. But we can go one step further in the simplification of the set of potential Steady State networks, and identify a single topology which constitutes the unique equilibrium network for a certain parameter's range. This topology essentially coincides with the real scientific collaboration network identified in the empirical studies mentioned above. Thus, our model naturally reproduces the scientific collaboration patterns observed in reality.

1.3 Literature Review

Theoretical social network formation models can be classified into two groups. On one hand, we have the physics-based modelling of society which treats agents as though they were just so much insensate matter (or rather, appearing perhaps to do so). That is, agents are non-strategic. This set has its origins in the random graph literature and has examples in the sociology literature and recently in the computer science and statistical physics literatures. References of this kind of models are abundant[5] but we will focus on two of them. Jackson and Rogers (06) proposes a nice, simple and general model of network formation. Authors combine random meeting and network-based meeting in a natural way and analyze how important are these two forces in determining the formation of different kinds of networks (scientific collaboration structures are one of them). The second model we want to emphasize presents a model that shares some features with the one we present here. Arenas et al (03) proposes a stylized model of a problem-solving organization –whose internal communication structure is given by a network– that can suffer congestion. Authors develop a design problem to determine which kind of network architectures optimizes performance for any given problem arrival rate. Contrarily to our model, network is fixed and players are non strategic.

Our work belongs to the other group of models. In this second set, models examine strategic formation of networks and use game theoretic tools. That is, there is no exogenous prescription of how the network is formed but there is a definition of the rules of the game agents have to play to form the network (See Jackson (2004) for a survey of this type of models). As introduced above, the work that more closely relates to our model is chapter 4 in Van der Leij (06). This author also tries to develop a theoretical model to explain the empirical regularities of research collaboration networks. In both models, heterogeneity across researchers plays a key role in explaining the results but contrarily to our paper, Van der Leij constructs a static model in which the link formation costs and the specific academic rewards scheme affect the equilibrium network topologies. Our model is dynamic and the possibility of congestion is the key factor (joint with agents' heterogeneity) for obtaining the results.

[5]See Newman (2003) for a survey. Some examples are Watts (1999), Cooper and Frieze (2003) or Price (1976).

2 General setting

Let N be the set of nodes, interpreted as researchers, with $n = |N|$ and let i and j be typical members of this set. We assume that n is finite and arbitrarily large. Networks are modeled as directed graphs. A directed graph on N is an $N \times N$ matrix g where entry g_{ij} indicates whether a directed link exists from node i to node j; $g_{ij} = 1$ indicates the existence of such a directed link and $g_{ij} = 0$ indicates the absence of this directed link. Notice that we do not impose any specific value for g_{ii}; in particular, it is possible to have $g_{ii} = 1$ (see interpretation below). For any node $i \in N$, let $N_i(g) = \{j \in N : g_{ji} = 1\}$ be the set of players that have a link towards i and $\eta_i(g) = |N_i(g)|$ denote the in-degree of i. On the other hand, let $M_i(g) = \{j \in N : g_{ij} = 1\}$ be the set of destinations of the links of i and $\mu_i(g) = |M_i(g)|$ denote the out-degree of i. Notice that $\eta_i(g)$ and $\mu_i(g)$ has to be natural numbers. We impose that $\mu_i(g) \geq 1$.[6]

The object of the agents of this model is to publish papers. This is their only source of utility. Specifically, a publication reports one unit of utility which will be equally split among all its coauthors. The starting point of a publication is an idea. At each point in time, modelled continuously, each researcher receives ideas from out of the network at an independent positive rate ρ. These ideas are *open*, in the sense that they need to be processed to become a publication. Immediately after receiving these open ideas, agents will send them to some previously selected destination (may be themselves). Here it is where the network plays its role, because a researcher i can only send her open ideas to some agent $j \in M_i(g)$ ($i \in M_i(g)$, i.e. $g_{ii} = 1$, means that agent i retains (part of) her own open ideas). We assume that all agents in $M_i(g)$ have exactly the same probability of being selected as destination of a particular open idea obtained by i. The node chosen as destination will be the researcher in charge of starting the publication process of this idea.

But at any point in time, several open ideas may "wait" to be processed by certain node (as in a queue) because we assume that researchers have a limited processing capability. Specifically, nodes process open ideas at a constant rate per instant of time, which we normalize to unity. Therefore, if a researcher receives a sufficiently high amount of collaboration requests (links), queues will be formed. Given this possibility, we must provide agents with some decision rule to select the open idea they will process from their stock. We will take the simplest one, that is, all open ideas in a queue have the same probability of being selected. Researchers also have a limited storage capability. In particular, each agent forgets an open idea with probability q at each point in time. For this reason, not all open ideas received by a node will be finally processed.

Once an open idea is chosen to be processed two things can happen: either it is published or it gets lost forever. Therefore, in this setting a publication can have at most two coauthors: the researcher who initially gets the open idea from out of the network and the destination of this open idea (notice that these two nodes can coincide). When an open idea is processed, the probability of being published by the coauthors (author) will depend on their (her) amount of talent. Let h be the vector of talent endowments and h_i be the i-th element of this vector interpreted as the agent i's amount of talent. We assume that h_i is exogenous, randomly generated following the probabilities

[6] when $\nexists j \neq i$ such that $g_{ij} = 1$, then g_{ii} must be necessarily 1. When $\exists j \neq i$ such that $g_{ij} = 1$, g_{ii} can also be 1.

described by any continuous distribution function[7] and that $h_i > 0$ for all $i \in N$. Vector h is fixed along the whole game. The relationship between talents and publication probability is determined by $f(\cdot)$. This is a strictly increasing probability function, holding $f(0) = 0$. This implies that the higher it is the amount of talent of a researcher/node the higher it is the probability of publishing the processed ideas. So, $f(h_i + h_j)$ is the probability of publishing a particular idea processed by i (or j) and previously sent by j (or i). Notice that h_i can also be interpreted as the agent i's productivity.

Therefore, at any point in time, agents are characterized by two defining features: an endogenous one, the size of their queue of open ideas waiting to be processed and an exogenous one, their amount of talent.

2.1 Network formation game and timing

At the beginning of any date, collaboration links are configured through the following network-formation game: all players $i \in N$ simultaneously announce the direct and directed links they wish to have either as origin or as destination. Formally, $S_i = \{0,1\}^{2n-1}$ is i's set of pure strategies. Let $s_i = (s^i_{i1}, s^i_{i2}, ..., s^i_{ii}, ..., s^i_{in}, s^i_{1i}, ..., s^i_{i-1,i}, s^i_{i+1,i}, ..., s^i_{ni}) \in S_i$. Then, $s^i_{ij} = 1$ if and only if player i wants to set up a directed link from i to j (and thus $s^i_{ij} = 0$, otherwise). As commented before $s^i_{ii} = 1$ is possible. A link, which is assumed to be costless, from player i to player j is formed if and only if $s^i_{ij} s^j_{ij} = 1$. That is, we assume that mutual consent is needed to create a link. Let $S = S_1 \times ... \times S_n$. A pure strategy profile $s = (s_1, ..., s_n) \in S$ induces a directed network $g(s)$.

Once the new network is formed, any agent (say i) receives open ideas at a rate ρ at each point in time and send them to one of her selected destinations. Simultaneously, node i selects and processes open ideas from her stock (if any) at a rate 1 per instant of time. Before ending the date, memory plays its role, so each open idea stored in the stock is forgotten with probability q. After all this process and just before the end of the period, the stock of open ideas of all nodes is updated.

2.2 Steady State analysis and payoff function

Suppose that the process reaches a Steady State and let us describe its characteristics. There are two defining properties of the Steady State: the stock of open ideas of the nodes is constant and the network is stable.

Let o_i be the Steady State stock of open ideas waiting to be processed by node i. Under stationarity, the number of open ideas standing in a queue behaves like a Markov process and the arrivals and departures from each node i follow Poisson processes. Given that in Steady State all open ideas that arrive to a node eventually depart from it in finite time, we must have that the arrival rate of open ideas must be equal to its departure rate. That is:

$$\rho \sum_{l \in N_i(g)} \frac{1}{\mu_l} = \begin{cases} 1 + q o_i & \text{, if } o_i \geq 1 \\ o_i(1+q) & \text{, otherwise} \end{cases} \quad \forall\, i \in N$$

[7]Notice that this implies that the probability of two agents having exactly the same amount of talent is zero.

The arrival rate of ideas to agent i is equal to the sum, over all nodes sending to i in g, of the expected number of ideas they get from out of the network per instant of time (ρ) times the probability of sending them to i. The departure rate is formed by the processing rate and the rate of open ideas that node i forgets. Notice that when the stock of open ideas is lower than one the processing capability of a node will be restricted. In such a case, only o_i open ideas can be processed per period (on average). From this expression we can write the Steady State stock of open ideas of a node as:

$$o_i = \begin{cases} \frac{\rho(\sum_{l\in N_i(g)} \frac{1}{\mu_l})-1}{q} & \text{, if } \sum_{l\in N_i(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho} \\ \frac{\rho(\sum_{l\in N_i(g)} \frac{1}{\mu_l})}{1+q} & \text{, otherwise} \end{cases} \quad \forall\, i \in N \tag{1}$$

As we can see, the stock of open ideas of a node is completely determined by the network structure (g), q and ρ.

The other defining feature of the Steady State is the stability of the network. But before defining the stability concept we introduce the payoff function. As commented above, researchers only obtain utility from the publication of ideas. Agent i's publications can derive from her stock of open ideas or from the open ideas that i previously sent to other researchers (notice that both sources can (partly) coincide when $g_{ii} = 1$). For a given network structure g, the following expression determines the expected payoff agent i obtains per period when the stock of open ideas is constant for all agents[8]:

$$\Pi_i(g) = \Theta(i)[\sum_{l\in N_i(g)\backslash i} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii}\frac{1}{\mu_i} f(h_i)] + \frac{1}{\mu_i}\sum_{l\in M_i(g)\backslash i} \Theta(l)cf(h_l+h_i) \text{ with } c = \frac{1}{2} \tag{2}$$

$$\text{where } \Theta(i) = \begin{cases} \frac{1}{\sum_{k\in N_i(g)}\frac{1}{\mu_k}} & \text{, if } \sum_{k\in N_i(g)} \frac{1}{\mu_k} \geq \frac{q+1}{\rho} \\ \frac{\rho}{1+q} & \text{, if } 0 < \sum_{k\in N_i(g)} \frac{1}{\mu_k} < \frac{q+1}{\rho} \\ 0 & \text{, if } \sum_{k\in N_i(g)} \frac{1}{\mu_k} = 0 \end{cases}.$$

For $g_{ji} = 1$, $\Theta(i)\frac{1}{\mu_j}$ can be interpreted as the steady state probability that an open idea coming from node j is chosen to be processed by i. The specific form of this probability derives from the assumption about the selection rule of the ideas of the queue (all ideas have the same probability of being selected). This probability is obtained by multiplying the share of ideas coming from j with respect to all ideas researcher i receives ($\frac{\frac{1}{\mu_j}}{\sum_{k\in N_i(g)}\frac{1}{\mu_k}}$) by the expected number of ideas that node i processes per period (1 if $\sum_{k\in N_i(g)} \frac{1}{\mu_k} \geq \frac{q+1}{\rho}$ and o_i otherwise[9]). About the payoff function, notice that there are three main factors affecting agent i's expected utility: c will influence the decision between retaining the open ideas or sending them to other authors, the queue size (included in $\Theta(l)$ or $\Theta(i)$) will affect the probability to process a specific open idea and the coauthor's amount of talent (h_l) will affect the probability of publishing the processed ideas.

The network stability concept used here is *pairwise-Nash Equilibrium*. In a PNE network, no player must have incentives to deviate unilaterally (that is the usual Nash Equilibrium condition)

[8]By using the per-period expected payoff to analyze the incentives to deviate from a particular network we do not consider the transition effects from one network to another. This simplification has minor implications specially for cases in which the transition does not last in time and/or the discounting rate is near to one.

[9]Notice that for a given g, q and ρ the expected stock of open ideas is determined by (1).

but we further require that any mutually beneficial link be formed in equilibrium. The pairwise-Nash Equilibrium networks are robust to bilateral commonly agreed one-link creation and to unilateral deviations. Formally, a pure strategy $s^* = (s_1^*, ..., s_n^*)$ is a Nash Equilibrium of the game of network formation (previously described) if and only if $\Pi_i(g(s^*)) \geq \Pi_i(g(s_i, s_{-i}^*))$, for all $s_i \in S_i$ and $i \in N$. Let $g + ij$ be the network obtained by adding the link g_{ij} to g.

Definition 1 *A network g is a pairwise-Nash equilibrium network with respect to the network payoff function Π if and only if there exists a Nash equilibrium strategy profile s^* that supports g, that is, $g = g(s^*)$, and, for all pair of players i and j such that $g_{ij} = 0$ if $\Delta\Pi_i(g + ij) > 0$ then $\Delta\Pi_j(g + ij) < 0$.*

3 Results

Our aim is to show whether this model can replicate some of the characteristics of real scientific collaboration networks. The empirical study Goyal, Van der Leij and Moraga (06) is an excellent basis for having a detailed image of the features of actual networks in Economics. The results of that paper *"show that the world of Economics is spanned by inter-linked stars, that this feature is stable over time and that this is the main reason for small average distances"*. From Newman (04) a similar conclusion could be extracted for Biology, Physics and Mathematics. In what follows, we will develop several results derived from the Steady State analysis of the model previously described that show how this empirical evidence can be a natural outcome of our setting. Specifically, we will see that for sufficiently low values of ρ a particular interlinked star network naturally arises –from the interaction of self-interested researchers– as the unique PNE structure. Heterogeneity and congestion are the key factors explaining these results.

In the following lines we present several propositions that identify certain features that PNE networks can not have for any arbitrarily low value of ρ. As a consequence, these results will allow us to narrow the set of potential PNE networks.

The first result refers to the number of out-degree links of a particular agent. In principle, the researchers of this model can send their open ideas to many collaborators, i.e. there is no upper bound on μ_i $\forall i$. The following result approaches this claim.

Let $G_{h,\rho}^*$ be the set of PNE networks for a given pair (h, ρ).

Proposition 1 *For any functional form of $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G_{h,\rho_0}^*$ in which $\mu_i > 1$ for some $i \in N$, there always exists a $\bar{\rho}_1 < \rho_0$ such that $g \notin G_{h,\rho}^*$ $\forall \rho < \bar{\rho}_1$.*

The proof (see in appendix) proceeds as follows. We consider all possible cases that a researcher i with $\mu_i \geq 2$ can face. Then, we analyze her incentives to sever the link with some of her collaborators. We show that, for any vector h, the marginal payoff derived from such a deviation tends to be positive as $\rho \to 0$. Using the concept of limit, we show how this is exactly a reformulation of the statement of the proposition. The intuition of the proof is very simple. The presence of heterogeneity implies that any agent can order the destinations of her open ideas with respect to the expected utility obtained from them. If an agent does not send all her open ideas to the destination that maximize this expected utility is because of another important factor: agents have limited processing capability. So congestion can arise and, as a consequence, queues can be formed. The longer it is the queue the lower it is the probability to process the ideas coming from a particular researcher. When an agent cuts some link off, she automatically increases the flow of open ideas sent to the rest of destinations and, as a consequence, increases their queue size if congestion arises. This has a negative impact on the expected utility of the agent who initially deviates. In the proof we show that as $\rho \to 0$ this negative impact tends to vanish. In consequence, for a sufficiently low value of ρ, any agent i with $\mu_i \geq 2$ will have incentives to deviate and send her open ideas to the unique destination that maximizes her expected payoff.

This result illustrates that whatever it is h, we can always find a sufficiently low value of ρ (say $\bar{\rho}_1$) such that a network in which some agent has two or more out-degree links cannot be sustained in equilibrium for any $\rho < \bar{\rho}_1$. Therefore, an upper bound on the equilibrium out-degree of nodes naturally arises for low values of ρ. As introduced above, this result implies a dramatic simplification of the set of possible PNE networks.

Focusing on the in-degree of nodes we can go one step further in that simplification. The following result establishes an upper bound for the amount of ideas a node can receive in a PNE network for any arbitrarily small ρ.

Proposition 2 *For any functional form of $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G^*_{h,\rho_0}$ in which $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ for some $i \in N$, there always exists a $\bar{\rho}_2 < \rho_0$ such that $g \notin G^*_{h,\rho}$ $\forall \rho < \bar{\rho}_2$.*

The intuition of the proof (see in appendix) is the following: when $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ for some $i \in N$, the stock of open ideas of i will be higher than one even after severing one in-degree link. Given that we have normalized the maximum processing rate to one, researcher i can delete one in-degree link without damaging the average processing flow (which will continue to be one). In consequence, agent i can increase her average productivity if she severs an in-degree link coming from a low-talent researcher who holds some specific conditions[10]. The proof shows that for any vector h, there exists a $\bar{\rho}_2$ that assures the existence of such a researcher for any $\rho < \bar{\rho}_2$.

The form of the production function has a direct effect on the incentives of collaboration. For a concave $f(\cdot)$, working with another researcher (rather than alone) increases the probability of publication less than proportionally with respect to the increase in the amount of talent. On the other hand, a convex $f(\cdot)$ implies that adding some additional talent in the production process increases the publication probability more than proportionally. For this reason, we can say that a concave $f(\cdot)$ discourages agents to look for collaborators. Previous results are valid for any functional form of $f(\cdot)$. But, in what follows, we will focus on the case in which $f(\cdot)$ is linear or convex. In that way we will specifically analyze the features of equilibrium networks in the cases in which collaborating with other researchers does not imply a loss of productivity with respect to working alone. For these cases, next result also narrows the set of potential PNE networks.

[10] In particular, this agent (say $j \in N_i$) must have a level of talent sufficiently low and must hold the following condition: $i \notin N_j$.

Proposition 3 *For a linear or convex $f(\cdot)$, a PNE network cannot have a player i with $\sum_{l \in N_i} \frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ when some agent j such that $h_j < h_i$ holds: $g_{jk} = 0$ $\forall k$ such that $h_k \geq h_i$.*

See proof in appendix. When $f(\cdot)$ is linear or convex and without considering the effects of a potential congestion, researchers always prefer to have collaborators with a higher talent. If $\sum_{l \in N_i} \frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ for some $i \in N$, agent i will not suffer congestion even after receiving a new link. For this reason, if all the researchers who receive ideas from some investigator j have a talent lower than h_i, agent j will have incentives to concentrate all the previous destinations of her ideas into a single one: agent i. Moreover, since i will not congest after receiving this new link, she will have incentives to form it because this only has a positive consequence: increasing her processing rate. Thus, such a pair of players cannot exist in a PNE network. Again, heterogeneity and congestion are essential for obtaining the result.

Notice that this proposition implies that high talent players have to receive a minimum flow of open ideas in any PNE network. Adding this result to Propositions 1 and 2 we can state the following:

Corollary 1 *For a linear or convex $f(\cdot)$ and for any vector h, high talent researchers have to receive all the links of the network in any $g \in G^*_{h,\rho}$ for any $\rho < \bar{\rho}_2$. Specifically, $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_i(g)} \frac{1}{\mu_l} < 1 + \frac{q+1}{\rho}$ for any high talent researcher i and any $\rho < \bar{\rho}_2$.*

The first claim of the corollary directly derives from Propositions 1 and 3. The bounds of $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$ are obtained in Propositions 2 and 3. This result illustrates a kind of attraction force of high-talent researchers in this model. This force is restricted by the possibility of congestion derived from the limited processing capability. For this reason, there is an upper bound in the number of open ideas a node can receive in equilibrium.

But notice that (for a linear or convex $f(\cdot)$) there is no heterogeneity requirement on the distribution of talents to obtain Proposition 3. So, this attraction force of the best players will act whatever it is the difference of their level of talent with respect to the rest. That is, the result holds for any vector h extracted from a continuous distribution function. On the other hand, since $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_i(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$, notice that the lower it is ρ the higher it is $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$. In consequence:

Corollary 2 *For a linear or convex $f(\cdot)$, in the PNE network candidates arbitrarily small differences among agents' productivity (talent) levels can generate highly unequal distributions of links. The lower it is ρ, the higher it is the in-degree inequality.*

Moreover, because of the convexity of the relationship between ρ and $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$, very small changes in ρ can translate into tremendous increments in the in-degree inequality among researchers. This contrasts with the results of Van der Leij (06) in which a minimum degree of heterogeneity among agents is required in order to reproduce the results we observe in reality.

So, we already have reproduced an empirical fact: in our Steady State network, some few agents can concentrate a lot of links (specially for low values of ρ). But, does this mean that we have an interlinked star in equilibrium as suggested by the empirical results of Goyal, Van der Leij and Moraga (06)? Following the previous results, that is not necessarily the case. For example, we can have a network formed by stars, in which the central agents (who receive a number of links respecting the bounds established in Corollary 1) are not connected between them. The next results focus on showing that this cannot happen for any arbitrarily small ρ. Therefore, interlinked stars must arise in equilibrium for low values of ρ. But in the next lines we go one step further and highlight a particular interlinked star network as a serious PNE candidate for any arbitrarily low ρ.

Let G^s denote the set of interlinked star networks holding the following three properties:

- $\mu_i = 1 \ \forall i \in N$.
- $g_{ii} = 1$ if and only if $h_i > h_l \ \forall l \in N$ and $l \neq i$.
- For any given pair (h_i, h_j) such that $h_i > h_j$ and $g_{ik} = g_{jl} = 1$ for any $k, l \in N$, it must be true that $h_k \geq h_l$.

The first condition simply states that all the nodes of the network have only one out-degree link. The second property implies that, except for the highest talent player, this link must go towards some other agent. Finally the last condition narrows the set of possible destinations of this out-degree link. To be clear the last two conditions imply that the agent with the highest talent (say agent 1) receives her in-degree links from the nodes located just below her in the ranking of talents; the second player in this ranking (agent 2) receives her in-degree links from the nodes located just below the first group of players in this ranking; and so on.

Lemma 1 *For any given pair (h, ρ) such that $\rho < \bar{\rho}_2$, there is a unique network in G^s in which no agent i with $\eta_i > 0$ has incentives to change η_i.*

See proof in appendix. Let g^s be the unique network derived from this lemma. Intuitively g^s is an interlinked star network in which the best researchers (the ones with a higher talent) receive their in-degree links from the best possible collaborators given what better researchers are receiving. Moreover, given Lemma 1, they do not have incentives to add (or delete) any in-degree link. The following two results show the relevance of g^s in this model.

Proposition 4 *For a linear or convex $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G^*_{h,\rho_0}$ different from g^s, there always exists a $\hat{\rho}_1 < \rho_0$ such that $g \notin G^*_{h,\rho} \ \forall \rho < \hat{\rho}_1$.*

See proof in appendix. This result shows that whatever it is h, we can always find a sufficiently low value for ρ under which a network different from g^s cannot be sustained as a PNE. Therefore, the set of potential PNE networks for any arbitrarily small ρ reduces to one single topology. In the proof we show that for any network $g \neq g^s$ we can always find a player (or a pair of players) whose marginal payoff for deviating tends to be positive as $\rho \to 0$. Notice that, using the definition of limit, this implies that such a network g cannot be PNE for any arbitrarily small ρ. Intuitively, the mechanisms underlying this result are the following ones. We first show that if we have a network different from g^s and $\rho < \bar{\rho}_1$, then there must exist an agent i who sends all her ideas to a node (say k) such that $h_k < h_j$, where j is the node that receives the ideas from i in g^s. If i does not deviate from this network and sends all her ideas to j is because of the possibility that j has a much longer queue than k due to congestion problems. As ρ goes to zero the differences between the queue sizes of different players (for example, j and k) become relatively smaller. Then, we reach a point in which the player i's incentives to deviate are basically driven by the differences between the talents of j and k. Since $h_k < h_j$, player i would have incentives to cut g_{ik} off and propose g_{ij}. For the deviation to take place, node j should have incentives to form the new link g_{ij}. In the worse case, agent j would only accept such a link if the average productivity from the ideas of her queue increases after the deviation. For a sufficiently low ρ, we can state that as $\rho \to 0$, this average productivity decreases. Then, there is a point in which this average is so low that the formation of g_{ij} push this average productivity up.

After Proposition 4, the set of PNE candidates reduces to one single network for an arbitrarily small ρ. The following result confirms that g^s is in fact a PNE for a sufficiently low ρ.

Proposition 5 *For a linear or convex $f(\cdot)$ and for any pair (h, ρ_0), if $g^s \notin G^*_{h,\rho_0}$ there always exists a $\hat{\rho}_2 < \rho_0$ such that $g^s \in G^*_{h,\rho}$, $\forall \rho < \hat{\rho}_2$.*

See proof in appendix. In the proof we check all the possible deviations from g^s. We find that, in the worse cases, the marginal payoff of potential deviators tends to be negative as $\rho \to 0$. Therefore for any vector h, we can always find a sufficiently low ρ such that no player have incentives to deviate from g^s. From an intuitive point of view, the proof can be explained as follows. There are two kinds of deviations from g^s. On the one hand, agents can substitute their out-degree links (only one per node) by a new one (or simply create an additional link) towards an agent with a talent lower than the one of the previous destination. In such a deviation, these agents trade-off the potential benefits from avoiding or reducing the effects of congestion against the costs of reducing the productivity of the processed ideas due to the lower talent of the new destination. In the proof we show that the positive part of this trade-off tends to vanish as $\rho \to 0$, and in consequence, the marginal payoff for deviating tends to be negative. The other type of deviation consists on substituting the current link by a new one (or simply create an additional link) towards an agent with a talent higher than the one of the previous destination. In that case, we use Lemma 1 to conclude that the marginal payoff of the new destination to accept the link will be negative.

From the last two propositions, immediately emerges the following corollary. Let $\hat{\rho} \equiv min(\hat{\rho}_1, \hat{\rho}_2)$.

Corollary 3 *For a linear or convex $f(\cdot)$ and for any vector h, there exists a $\hat{\rho}$ such that g^s is the unique PNE network for any $\rho < \hat{\rho}$.*

Existence of g^s as a PNE comes from Proposition 5 and uniqueness comes from Proposition 4.

Notice again, that we do not need to impose any degree of heterogeneity among agents. Specifically any talent vector h extracted from a continuous distribution function can generate the previous result. For these reasons we conclude that the kind of networks GVM observes in reality are a natural outcome from the interaction of self-interested researchers of this model.

Before ending this section of results, it might be interesting to give some hints about the behavior of the model for a concave $f(\cdot)$. As commented above, concavity of $f(\cdot)$ discourages agents at the moment of looking for collaborations. In consequence, we cannot assure the stability of the interlinked star for all distributions of talent even for arbitrarily small ρ. In particular, we would need a sufficiently high inequality across levels of talent to support such a network of collaborations. This inequality is also required to support other structures with high in-degree inequality (non-interlinked stars) as stable networks.

Remark 1 *For $f(\cdot)$ concave, networks with high in-degree inequality (as an interlinked star) may not be PNE even for arbitrarily small ρ. To assure stability of this kind of networks we need a minimum degree of heterogeneity among researchers' talents.*

On the other hand, other networks such as the empty network or the cycles can arise in equilibrium even with low values of ρ, specially when inequality across levels of talent is not so high. Summarizing, for a concave $f(\cdot)$ the key factor affecting the shape of the stable network would be the inequality across levels of talent. Only highly unequal talent distributions will allow to obtain stable networks with a high in-degree inequality such an interlinked star.

4 Discussion

4.1 Empirical patterns

To reach the conclusion that the world of Economics is spanned by interlinked stars Goyal, Van der Leij and Moraga (06) analyzes some empirical patterns. In this paper we showed that a very simple network formation model characterized by the limited processing capability of heterogeneous agents can reproduce the features of the in-degree distribution of the so called interlinked star network in equilibrium. In this section, we discuss how can our model be extended to explain some other empirical patterns.

One of the first empirical findings of GVM refers to the average number of collaborators. For the giant component of the analyzed coauthorship network this average goes from 2.48 in the 1970's until 3.06 in the 1990's[11]. None of the results of our model excludes the possibility of having these average numbers of collaborators. In fact, we can have equilibrium networks with 2, 3 or more collaborators per researcher. But, as Proposition 1 shows for low values of ρ, it would be specially difficult to have some researcher $i \in N$ with more than one out-degree link in a PNE network. Therefore, for low values of ρ our model can hardly reproduce this average number of collaborators.

A very easy extension of the model will allow us to naturally reproduce this empirical fact. Imagine that there exist different types of talent. Researchers are specialists, so they have a specific type. Moreover, ideas can require some specific type of talent to be published that does not necessarily coincide with the type of talent of the first receiver. For this reason, we can also classify the ideas on

[11] Newman (04) founds that the average number of collaborators in Biology, Physics and Mathematics were 18.1, 9.7 and 3.9 respectively.

different types. To be specific let h_i^x denote the agent i's amount of talent of type x. Let $f_x(h_i^x + h_j^x)$ be the expected probability of publishing a processed idea of type x by researchers i and j. The rest of the model would not change with respect to the one described in section 2. In this new model, agents would have incentives to select specialist collaborators for each of the different types of ideas they can receive. Thus, equilibrium networks would be able to reproduce higher average numbers of collaborators. Moreover the average number of collaborators in equilibrium would positively depend on the degree of researchers' specialization. Therefore the model would constitute a formalization of the argument defending that one of the key factors explaining the increase in the flow of scientific collaboration is the increase in the specialization of researchers. This increase in the number of collaborators is a trend that GVM detected for the last 30 years in the world of Economics.

With respect to the degree distribution, GVM founds that such a distribution exhibits fat-tails with a small fraction of scientists having a large number of collaborators. The same can be concluded for the fields of Biology, Physics and Mathematics as Newman (04) shows. The results of our model show that the links concentrate in the high-talent researchers. Decreasing the value of ρ will increase the number of links directed towards each of these researchers and, in consequence, increase the inequality in the in-degree distribution. At this point it is worth to mention a particularity of our model. In equilibrium high talent researchers will roughly have the same number of links (see Corollary 1). Evidently, this is not the case in actual networks. This result arises because of a very easy simplification of the model. We assume that all players in our game have exactly the same processing capability which we have normalized to 1. By allowing different processing rates, the model would be able to reproduce equilibria with different in-degree levels for different players.

The last empirical pattern we want to discuss is referred to clustering. GVM shows that *"the most connected individuals collaborated extensively and most of their coauthors did not collaborate with each other"*. Our model cannot give an intuitive explanation for that. Focusing on the role of heterogeneity among players and their limited processing capability (as our model does), we can only give an intuitive argument for explaining a lower or higher number of links per node which is not a sufficient condition for explaining their level of clustering. In order to explain the patterns of clustering, probably we have to look for arguments related with some kind of geographic or conceptual proximity between researchers. A very simple extension of the model can capture this kind of considerations. Let us assume that researchers are distributed in groups. A group can be defined in a very broad sense. Two researchers can be members of the same group if they are in the same department, if they work on similar topics or if they share a common personal characteristic. If two researchers i and j are in the same group then $d_{ij} = 1$; otherwise $d_{ij} = 0$. We can reasonably argue that two members of the same group will have more facilities to collaborate with each other and this will increase their joint productivity. Formally we can write the expected probability of publishing a processed idea by nodes i and j as $f(h_i + h_j + kd_{ij})$ for some $k > 0$. This simple extension will allow the model to reproduce equilibrium networks with a high clustering between members of the same group and low clustering between the high talent researchers of different groups. This kind of a priory distributions of agents in a network formation model are analyzed by Rubí-Barceló

(07)[12].

4.2 Stability and efficiency

A network is efficient when it maximizes the aggregate payoff. The set of efficient networks usually does not coincide with the set of stable networks. In fact, one of the most habitual analysis in the network formation models is the comparison between the stable and the efficient networks of the model. Jackson (04) collects a great variety of examples of this kind of studies. The most similar to ours appears in Jackson and Wolinsky (96). These authors develop a very simple coauthorship network formation model as an example of how negative externalities can play an important role in the network formation process of the academic world, and in particular, in the conflict between stability and efficiency. In the efficient network of the model, researchers are distributed in pairs, i.e. two agents connected with each other and isolated to the rest. But, the stable network is over connected with respect to the efficient one. In that model, the good strategy from an individualistic point of view does not coincide with the good strategy from an aggregate point of view. This problem, is specially relevant when talking about coauthorship networks because it implies that researchers' individual incentives are damaging the total scientific production. Therefore, the example of Jackson and Wolinsky (96) offers a very pessimistic image of what could happen in actual scientific collaboration networks. In the next lines we will show that in our model, individual and aggregate incentives are much more aligned. In spite of that, they do not fully coincide.

The results of Section 3 clearly point out towards g^s as the stable network for any arbitrarily small ρ when $f(\cdot)$ is linear or convex. In this section we will look for the efficient network(s) in this case. This will make the comparison between stability and efficiency easier. A priory, we can already say that g^s has favorable features to maximize the aggregate payoff. In particular, for a linear or convex $f(\cdot)$ seems to be suitable that high talent players collaborate with each other. But, is the structure of collaborations of g^s the best one in order to maximize the aggregate payoff for any given pair (h, ρ)? The next result answers to this question for an arbitrarily small ρ.

Proposition 6 *For $g = g^s$ and for any vector h, there exists a ρ^* such that for any $\rho < \rho^*$, if we substitute a link g_{ij} by a new link g_{ik} then the marginal aggregate payoff decreases when $h_k < h_j$ and increases when $h_k > h_j$.*

See proof in appendix. That is, the accumulation of links towards the high talent players have positive implications for the aggregate payoff when ρ is sufficiently small. Once again the trade off between the benefits of working with high talent researchers and the costs of working with more congested coauthors come on stage. By changing g_{ij} by a new link g_{ik} such that $h_k < h_j$, the ideas of agent i can avoid congestion problems but they will have a lower publication probability once processed. On the other hand, by changing g_{ij} by a new link g_{ik} such that $h_k > h_j$, k can suffer congestion problems but i increases the publication probability of her processed ideas. But when $\rho \to 0$ the congestion problems tend to disappear. As a consequence, the aggregate marginal payoff

[12] In that paper the payoff function is very different to the one used here.

increases if high talent players receive more ideas and decreases when high talent players receive less ideas.

The result show that the PNE interlinked star network we obtained in the previous section is not efficient for any ρ. And also shows that the way of increasing efficiency when the entrance rate of open ideas is sufficiently low, is accumulating links towards the high talent researchers.

4.3 Structural Holes and heterogeneity

The notion of Structural Hole in social networks was first introduced by Burt (92). A Structural Hole is a disconnection among agents on the network structure. Several authors[13] provide empirical evidence that people who bridge structural holes in social networks have significantly higher payoffs. In particular, Burt (2004) shows, in a firm environment, that compensation, positive performance evaluations, promotions and good ideas are disproportionately in the hands of people whose networks span structural holes.

To explain this empirical evidence one can follow two opposite lines of reasoning. First, one can say that is because of the position in the network that players can enjoy a significantly higher payoff. The theoretical model of Goyal and Vega (07), will reinforce this reasoning. In that model, the authors show that ex-ante identical agents configure a stable network in which there are very high inequalities in the equilibrium payoff distribution –a single player obtains a disproportionately higher payoff due to her position.

On the other hand, one can defend that ex-ante differences between agents explain the differences in the location and, in consequence, in the payoff these agents can reach in equilibrium. This is the line of reasoning our paper follows since what finally generates that good players become the center of the stars in the final equilibrium networks is the assumed heterogeneity among agents.

5 Conclusion

In spite of the great variety of empirical papers about scientific collaboration networks, there is a lack of foundational theoretical models that analyze how the decisions of individuals contribute to scientific collaboration network formation. This paper proposes a dynamic model to analyze the formation of this kind of networks.

We focus on the role of heterogeneity among agents' productivity and congestion problems derived from their limited processing capability to show that self-interested researchers can organize themselves in the way we observe in reality, which is, inter-linked stars.

[13]Burt (1992), Mehra, Kilduff and Bass (2003), Podolny and Baron (1997), Ahuja (2000)

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A Proofs

Proof of Proposition 1. Let g be a PNE for a given pair (h, ρ_0). Let $i \in N$ be a member of this network who has $\mu_i \geq 2$. We claim that there exists a $\bar{\rho}_1 < \rho_0$ such that g will not be PNE for any pair (h, ρ) such that $\rho < \bar{\rho}_1$.

Let us focus on the case in which $\mu_i = 2$ (the cases in which $\mu_i > 2$ can be proved analogously). Let players k and j be the destinations of these two links, that is, $g_{ij} = g_{ik} = 1$. Here we consider the case in which $i \neq j$ and $i \neq k$. The case in which $i = j$ or $i = k$ is analogous, thus omitted. Next we analyze the i's incentives to deviate. At this point we have to distinguish the following cases:

i $\sum_{l \in N_r(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ for $r = k, j$.

The i's marginal payoff for cutting the link g_{ij} off is positive if and only if:

$$\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}\left[\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l}} + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}\right] \tag{i}$$

On the other hand, the i's marginal payoff for cutting the link g_{ik} off is positive if and only if:

$$\frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}\left[\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l}} + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}\right] \tag{ii}$$

Given that $0 < \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}} \leq \frac{\rho}{q+1}$ for $r = k, j$ we can say that:

$$lim_{\rho \to 0}\left(\frac{1}{\sum_{l \in N_r} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}}\right) = 0$$

Following the definition of limit we can say that for any $\varepsilon > 0$, there exists a ρ' such that $\left|\frac{1}{\sum_{l \in N_r} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}}\right| < \varepsilon$ for $r = k, j$, for any $\rho < \rho'$. Given that $h_j \neq h_k$, notice that if the LHS of conditions (i) and (ii) were equal to $\frac{f(h_i + h_r)}{\sum_{l \in N_r} \frac{1}{\mu_l}}$ for $r = k, j$ respectively, (i) and (ii) would be complementary. So, we can conclude that for any specific vector h, and in particular for any specific triple (h_i, h_j, h_k), there exists a ρ' such that some of the two conditions (i) and (ii) has to hold for any $\rho < \rho'$. That is, for a sufficiently small ρ agent i will have incentives to deviate and cut one of her out-degree links off.

ii $\sum_{l \in N_j(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ and $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l \in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ (or vice versa).

The i's marginal payoff for cutting the link g_{ij} off is positive if and only if:

$$\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}\left[\frac{\rho}{1+q} f(h_i + h_k) + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}\right] \tag{iii}$$

The i's marginal payoff for cutting the link g_{ik} off is positive if and only if:

$$\frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}\left[\frac{\rho}{1+q} f(h_i + h_k) + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}\right] \tag{iv}$$

Given that $0 < \frac{1}{\sum_{l\in N_r(g)} \frac{1}{\mu_l}+\frac{1}{2}} \leq \frac{\rho}{q+1}$ for $r = k, j$ we can say that $lim_{\rho\to 0}(\frac{1}{\sum_{l\in N_k} \frac{1}{\mu_l}+\frac{1}{2}} - \frac{\rho}{1+q}) = 0$ and $lim_{\rho\to 0}(\frac{1}{\sum_{l\in N_j} \frac{1}{\mu_l}+\frac{1}{2}} - \frac{1}{\sum_{l\in N_j} \frac{1}{\mu_l}}) = 0$. Following the definition of limit we can say that for any $\varepsilon > 0$, there exists a ρ'' such that $|\frac{1}{\sum_{l\in N_k} \frac{1}{\mu_l}+\frac{1}{2}} - \frac{\rho}{1+q}| < \varepsilon$ and $|\frac{1}{\sum_{l\in N_j} \frac{1}{\mu_l}+\frac{1}{2}} - \frac{1}{\sum_{l\in N_j} \frac{1}{\mu_l}}| < \varepsilon$ for any $\rho < \rho''$. Given that $h_j \neq h_k$, notice that if the LHS of conditions (iii) and (iv) were equal to $\frac{\rho f(h_i+h_k)}{1+q}$ and $\frac{f(h_i+h_j)}{\sum_{l\in N_j} \frac{1}{\mu_l}}$ respectively, (iii) and (iv) would be complementary. So, we can conclude that for any specific vector h, and in particular for any specific triple (h_i, h_j, h_k), there exists a ρ'' such that some of the two conditions (iii) and (iv) has to hold for any $\rho < \rho''$. That is, for a sufficiently small ρ agent i will have incentives to deviate and cut one of her out-degree links off.

iii $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l\in N_r(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ for $r = k, j$.

iv $\sum_{l\in N_j(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ and $\sum_{l\in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$.

v $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l\in N_j(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ and $\sum_{l\in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$.

The proof for these three cases proceeds analogously to the previous one.

vi $\sum_{l\in N_r(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$ for $r = k, j$.

In this case it is easy to check that agent i will have incentives to sever the link with the agent with the lowest level of talent for any given ρ.

Therefore, in any possible case in which $\exists i \in N$ such that $\mu_i \geq 2$, we can find a sufficiently low value for ρ under which there is a profitable deviation. Defining $\bar{\rho}_1$ as the minimum of all these values of ρ, the proof of the proposition is done. ■

Proof of Proposition 2. Imagine we have an agent i such that $\sum_{l\in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ in a network $g \in G^*_{h,\rho_0}$ for a given pair (h, ρ_0). First we claim that there is a ρ (say $\bar{\rho}_2$) such that for any $\rho < \bar{\rho}_2$ the talents of the members of N_i hold one of the following two sets of conditions:

a $\exists k \in N_i$ such that $k \notin M_i$ who holds:

$$cf(h_i + h_k) < \frac{1}{\sum_{l\in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}\left(\sum_{l\in N_i\setminus\{i,k\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii}\frac{1}{\mu_i} f(h_i)\right) \quad \text{(v)}$$

b $g_{ii} = 1$ and

$$f(h_i) < \frac{1}{\sum_{l\in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_i}} \sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l} cf(h_l + h_i)$$

Let us proof this initial claim. Since the levels of talent are extracted from a continuous distribution function, either we are in case (b) or there must exist some agent $k \in N_i$ $(k \neq i)$ whose joint productivity with i $(cf(h_i + h_k))$ is below the average productivity of the rest of agents of N_i, i.e. some $k \in N_i$ $(k \neq i)$ holds condition (v). But, for (a) to be hold this agent k should not receive a link from i. In the next lines we show that for a sufficiently low value of ρ such an agent should exist.

Notice that from Proposition 1, there exists a ρ (that we call $\bar{\rho}_1$) such that $\mu_i = 1$ in any $g \in G^*_{h,\rho}$ for any $\rho < \bar{\rho}_1$. Therefore, if at least two agents $j, k \in N_i$ have a joint productivity with i below the average of the rest of players in N_i, we already know that one of them will not be in M_i for any $\rho < \bar{\rho}_1$. But there can be a single agent (say j) such that $j \in M_i$ and whose joint productivity with i is below the average of the rest of players in N_i. Next, we show that for a sufficiently low ρ this case cannot be hold when $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho}$.

Imagine that $cf(h_j + h_i) = \varepsilon$ for $\varepsilon > 0$ arbitrarily small. Let h_k be such that:

$$\begin{cases} f(h_j + h_i) < f(h_k + h_i) < f(h_l + h_i) \ \forall l \in N_i \backslash \{i\} \text{ and } cf(h_k + h_i) < f(h_i) & \text{for } k \neq i \\ f(h_k) < cf(h_l + h_i) \ \forall l \in N_k \backslash \{k, j\} & \text{for } k = i \end{cases}$$

Let us assume that $cf(h_k + h_i) = b$ for $k \neq i$ (or $f(k) = b$ for $k = i$). We know that there is an $\varepsilon > 0$ arbitrarily small under which $cf(h_l + h_i) > b + \varepsilon \ \forall l \in N_i \backslash \{i, j, k\}$.Then, the average productivity of the players of N_i different from k is higher than:

$$\frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} (\varepsilon + (\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_j} - \frac{1}{\mu_k})(b + \varepsilon))$$

After some simple algebra we can conclude that this expression is higher than b when the following condition holds:

$$b + \varepsilon > (b + \frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}) \frac{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k} - \frac{1}{\mu_j} + 1}$$

Given that $\mu_j \geq 1$, $\frac{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k} - \frac{1}{\mu_j} + 1} \leq 1$. Since $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho}$, we can conclude that $lim_{\rho \to o} \frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} = 0$. This implies that for any $\varepsilon > 0$, we can always find a value of ρ (say ρ') such that $|\frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}| < \varepsilon$ for any $\rho < \rho'$. Therefore, for any $\rho < \rho'$ we have at least two agents (j and k) who hold condition (v). As commented above, one of them must fulfill all the conditions of case (a) when $\rho < \bar{\rho}_1$. Let $\bar{\rho}_2 \equiv min(\bar{\rho}_1, \rho')$. That concludes the proof of the initial claim.

Next step is to show that a network that holds one of these two cases cannot be sustained as a PNE for $\rho < \bar{\rho}_2$.

If we have case (a), agent i's marginal payoff for cutting the link g_{ki} off is:

$$\Delta \Pi_i = \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} [\sum_{l \in N_i \backslash \{i,k\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)] - \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l}} [\sum_{l \in N_i \backslash \{i\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)]$$

After some simple algebra we can say that $\Delta \Pi_i > 0$ if and only if condition (v) holds. This happens by definition of k. Thus, agent i have incentives to deviate.

If we have case (b), agent i's marginal payoff for changing the link g_{ii} for a new link $g_{ij} \ \forall j \notin N_i$ is:

$$\triangle \Pi_i = \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l}} [\frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_i}} \sum_{l \in N_i \backslash \{i\}} \frac{1}{\mu_l} cf(h_l + h_i) - f(h_i) + \theta_k \eta_i cf(h_k + h_i)]$$

where $\theta_k = \begin{cases} \frac{\rho}{1+q} & \text{if } \frac{(\sum_{l\in N_k}\frac{1}{\mu_l}+\frac{1}{\mu_i})\rho}{q+1} < 1 \\ \frac{1}{\eta_k+1} & \text{otherwise} \end{cases}$.

Under conditions of case (b), $\frac{1}{\sum_{l\in N_i}\frac{1}{\mu_l}-\frac{1}{\mu_i}} \sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l} cf(h_l+h_i) > f(h_i)$, thus $\triangle\Pi_i > 0$. For the deviation to take place, agent j should accept the formation of the link g_{ij}. Given Proposition 1, we can say that there exists a value of ρ (say $\bar{\rho}_1$) such that $\mu_l = 1$ for all $l \in N$, for any $\rho < \bar{\rho}_1$. Given that $\sum_{l\in N_i}\frac{1}{\mu_l} \geq 1+\frac{q+1}{\rho}$, we can say that $\eta_i \geq 2$. Since $\mu_l = 1$ for all $l \in N$, there must exist some agent $k \in N$ with $\eta_k = 0$. For such a node the marginal payoff for accepting the link is:

$$\Delta\Pi_k \geq cf(h_k+h_i) > 0$$

Notice that $\Delta\Pi_k > cf(h_k+h_i)$ only when $g_{ki} = 1$. Therefore, both players i and k have incentives to deviate and form the link g_{ij}.

Given that $\bar{\rho}_2 \leq \bar{\rho}_1$, the proof is completed. ■

Proof of Proposition 3. Before the proof of this proposition we need an additional lemma.

Lemma 2 *If $c \geq \frac{1}{2}$, $f(0) = 0$ and $f(\cdot)$ is linear or convex, then $cf(h_k+h_l) > f(h_l)$ for $h_k > h_l$.*

Proof. *For a linear or convex $f(\cdot)$ with $f(0) = 0$ and $c = \frac{1}{2}$, the inequality $cf(h_k+h_l) > f(h_l)$ reduces to $h_k > h_l$ after some simple algebra. For $f(\cdot)$ convex and $c > \frac{1}{2}$, the difference between $cf(h_k+h_l)$ and $f(h_l)$ will be higher; therefore, the inequality of the statement also holds.* ■

By contradiction let us assume that we have a PNE network in which agent i holds $\sum_{l\in N_i}\frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ and there exists an agent j with $h_j < h_i$ who holds: $g_{jk} = 0$ $\forall k$ such that $h_k \geq h_i$. Imagine the deviation in which player j cuts all her out-degree links and proposes to agent i the formation of a new one. In such a case the marginal utility for player j is:

$$\Delta\Pi_j = \frac{\rho}{1+q} cf(h_j+h_i) - \frac{1}{\mu_j}\Big[\sum_{l\in M_j\setminus\{j\}} \Theta(l) cf(h_l+h_j) + g_{jj}\Theta(j) f(h_j)\Big]$$

From definition we know that $\Theta(l)$ and $\Theta(j)$ are lower or equal than $\frac{\rho}{1+q}$. By assumption, $h_l < h_i$ $\forall l \in M_j$. On the other hand, by Lemma 2 we can say that for a linear or convex $f(\cdot)$ and $c \geq \frac{1}{2}$, $cf(h_j+h_i) > f(h_j)$. All that restrictions imply that $\Delta\Pi_j > 0$.

But in order to complete the proof we need to show that agent i will have incentives to form the link. Her marginal utility from accepting it is:

$$\begin{aligned}\Delta\Pi_i = \frac{\rho}{1+q}\Big[&\sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l} cf(h_i+h_l) + g_{ii}\frac{1}{\mu_i} f(h_i) + cf(h_i+h_j)\Big] \\ &+ \frac{c}{\mu_i} \sum_{\substack{l\in M_i\setminus\{i\} \\ l\in M_j}} \hat{\Theta}(l) cf(h_l+h_i) \\ &- \Big[\frac{\rho}{1+q}\Big[\sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l} cf(h_i+h_l) + g_{ii}\frac{1}{\mu_i} f(h_i)\Big] + \frac{c}{\mu_i} \sum_{\substack{l\in M_i\setminus\{i\} \\ l\in M_j}} \Theta(l) cf(h_l+h_i)\Big]\end{aligned}$$

where $\hat{\Theta}(l)$ corresponds to the variable $\Theta(l)$ for the new situation in which agent j has cut all her out-degree links off. After some simple algebra:

$$\Delta\Pi_i = \frac{\rho}{1+q} cf(h_i + h_j) + \frac{c}{\mu_i} \sum_{\substack{l \in M_i \setminus \{i\} \\ l \in M_j}} (\hat{\Theta}(l) - \Theta(l)) cf(h_l + h_i)$$

For $l \in M_j$, $\hat{\Theta}(l) > \Theta(l)$. Therefore $\Delta\Pi_i > 0$, contradicting the initial statement of stability. ■

Proof of Lemma 1. In this proof we will show that for any given pair (h, ρ) such that $\rho < \bar{\rho}_2$, if $g \in G^s$ there is only one possible distribution of links under which no player have incentives to change her amount of in-degree links. This implies that only one network in G^s can hold this condition. To demonstrate it, we proof that to not having incentives to change η_i, any node must have a given amount of in-degree links.

Imagine a player i and a given in-degree $\eta_i > 0$. Given that $\rho < \bar{\rho}_2 \leq \bar{\rho}_1$, players can have at most a single out-degree link in any PNE network (Proposition 1). Then, $\sum_{l \in N_i} \frac{1}{\mu_l} = \eta_i \; \forall i \in N$. Moreover since $\rho < \bar{\rho}_2$, we can say that $\frac{q+1}{\rho} - 1 \leq \eta_i < \frac{q+1}{\rho} + 1$ (Corollary 1). Given that η_i can only take natural numbers, there are at most two possible values for η_i in any PNE (say $\bar{\eta}$ and $\bar{\eta} - 1$). We claim that for only one of these two values agent i will not have incentives to change her in-degree for a given pair (h, ρ). First of all, it is easy to check that the following must hold:

$$\frac{q+1}{\rho} - 1 \leq \bar{\eta} - 1 < \frac{q+1}{\rho} \leq \bar{\eta} < \frac{q+1}{\rho} + 1$$

Given these inequalities, it is easy to conclude that in any $g \in G^s$ no player i with $\eta_i = \bar{\eta}$ will have incentives to accept an additional in-degree link. Moreover, no player i with $\eta_i = \bar{\eta} - 1$ will have incentives to delete one existing in-degree link. Therefore, there are only two possibilities of deviation. An agent i can have incentives to cut one in-degree link off when $\eta_i = \bar{\eta}$ or she can have incentives to accept some additional in-degree link when $\eta_i = \bar{\eta} - 1$. Analyzing the marginal payoff of both deviations, we will observe that if one is positive the other must be negative.

Let $\eta_i = \bar{\eta}$ and $g_{ji} = 1$ in a given network g_1.The agent i's marginal payoff for deleting the in-degree link g_{ji} is:

$$\Delta\Pi_i = \frac{\rho}{1+q} [\sum_{l \in N_i(g_1) \setminus \{i,j\}} cf(h_i + h_l) + g_{ii} f(h_i)] - \frac{1}{\bar{\eta}} [\sum_{l \in N_i(g_1) \setminus \{i\}} cf(h_i + h_l) + g_{ii} f(h_i)]$$

After some simple algebra, $\Delta\Pi_i > 0$ if and only if:

$$(\frac{\bar{\eta}\rho}{1+q} - 1)[\sum_{l \in N_i(g_1) \setminus \{i,j\}} cf(h_i + h_l) + g_{ii} f(h_i)] > cf(h_i + h_j)$$

On the other hand, let $\eta_i = \bar{\eta} - 1$ and $g_{ji} = 0$ in a given network g_2. The agent i's marginal payoff for creating the link g_{ji} is:

$$\Delta\Pi_i = \frac{1}{\bar{\eta}} [\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) + cf(h_i + h_j) + g_{ii} f(h_i)] - \frac{\rho}{1+q} [\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) + g_{ii} f(h_i)]$$

After some simple algebra, $\Delta\Pi_i > 0$ if and only if:

$$(\frac{\bar{\eta}\rho}{1+q}-1)[\sum_{l\in N_i(g_2)\backslash\{i\}} cf(h_i+h_l)+g_{ii}f(h_i)] < cf(h_i+h_j)$$

Notice that $\sum_{l\in N_i(g_2)\backslash\{i\}} cf(h_i+h_l) = \sum_{l\in N_i(g_1)\backslash\{i,j\}} cf(h_i+h_l)$. Therefore, one (and only one) of the two previous inequalities will hold for any vector h. In consequence, in one (and only one) of the amounts of in-degree links player i will not have incentives to change her η_i. ■

Proof of Proposition 4. Let g be a PNE network for some pair (h,ρ_0), i.e. $g \in G^*_{h,\rho_0}$. Let g^s_{ij} denote the link g_{ij} in the network g^s ($g^s_{ij}=1$ if and only if node i have a link towards j in g^s). Imagine that g is different from g^s. This implies that there exists an agent (say i) such that $g_{il} \neq g^s_{il}$ for some $l \in N$. For this to be the case we have several possibilities. In the next lines we will show that there always exists a sufficiently low value of ρ (say $\hat{\rho}_1$) under which none of these possibilities can be sustained in a PNE for any $\rho < \hat{\rho}_1$. Let $j \in N$ be such that $g^s_{ij}=1$.

(a) The first possibility is that agent i has more than one out-degree link. Following Proposition 1, we know that there exists a ρ (we called $\bar{\rho}_1$) such that such a network cannot be sustained as a PNE for any $\rho < \bar{\rho}_1$.

(b) Second, agent i can have a link towards k, i.e. $g_{ik}=1$, and $h_k < h_j$. Now let us show that this second case cannot be hold in a PNE network. Two subcases need to be considered:

- $k=i$. In such a case, consider that player i substitutes the link g_{ii} by the new link g_{ij}. Let consider the extreme (and less favorable) case in which $o_i \leq 1$ and $\sum_{l\in N_j}\frac{1}{\mu_l} \geq \frac{q+1}{\rho}$. The agent i's marginal payoff from this deviation is:

$$\Delta\Pi_i = \frac{c}{\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i}}\frac{1}{\mu_i}f(h_j+h_i) + \frac{\rho}{1+q}[\sum_{l\in N_i\backslash\{i\}}\frac{1}{\mu_l}cf(h_l+h_i)] - \frac{\rho}{1+q}[\sum_{l\in N_i\backslash\{i\}}\frac{1}{\mu_l}cf(h_l+h_i)+\frac{1}{\mu_i}f(h_i)]$$

We can conclude that $\Delta\Pi_i > 0$ if and only if:

$$cf(h_j+h_i) > \frac{\rho(\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i})}{1+q}f(h_i) \quad \text{(vi)}$$

Given Proposition 2, $\exists\bar{\rho}_2$ such that $\sum_{l\in N_j}\frac{1}{\mu_l} < 1+\frac{q+1}{\rho}$ for any $\rho < \bar{\rho}_2$ in a PNE. Given that $\sum_{l\in N_j}\frac{1}{\mu_l} \geq \frac{q+1}{\rho}$, we can write that for any $\rho < \bar{\rho}_2$, $\frac{\rho(\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i})}{1+q} \in [1+\frac{\rho}{1+q}\frac{1}{\mu_i}, 1+\frac{\rho}{1+q}(1+\frac{1}{\mu_i}))$. So, $lim_{\rho\to 0}\frac{\rho(\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i})}{1+q} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j+h_i) > f(h_i)$ for $f(\cdot)$ linear or convex and $c \geq \frac{1}{2}$. Following the definition of limit, we can conclude that for a linear or convex $f(\cdot)$ and for any $\varepsilon > 0$, there always exists a $\rho' \leq \bar{\rho}_2$ such that $|\frac{\rho(\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i})}{1+q} - 1| < \varepsilon$, $\forall \rho < \rho'$. Therefore, for any vector h, and in particular for any pair (h_j, h_i) we can find a sufficiently low value of ρ (say ρ') such that $\frac{\rho(\sum_{l\in N_i}\frac{1}{\mu_l}+\frac{1}{\mu_{i+1}})}{1+q}$ is sufficiently close to 1 to hold condition (vi) for any $\rho < \rho'$ (when $f(\cdot)$ is linear or convex).

– $k \neq i$. Let us assume the extreme (and less favorable) case in which $o_k < 1$ and $\sum_{l\in N_j}\frac{1}{\mu_l} \geq \frac{q+1}{\rho}$. The marginal payoff derived from the change of destination of open ideas of agent i to agent j is positive when:

$$f(h_j + h_i) > \frac{(\sum_{l\in N_j}\frac{1}{\mu_l} + \frac{1}{\mu_i})\rho}{1+q} f(h_i + h_k) \tag{vii}$$

From the previous point we already know that for $\rho < \bar{\rho}_2$, $lim_{\rho\to 0}\frac{\rho(\sum_{l\in N_j}\frac{1}{\mu_l}+\frac{1}{\mu_i})}{1+q} = 1$. Given that in case (b) $h_j > h_k$, following the definition of limit we can conclude that there always exists a $\rho'' < \bar{\rho}_2$ such that $\frac{\rho(\sum_{l\in N_i}\frac{1}{\mu_l}+\frac{1}{\mu_{i+1}})}{1+q}$ is sufficiently close to 1 to hold condition (vii) for any $\rho < \rho''$.

(c) Notice that there is a third case in which i has a unique link towards k and $h_k > h_j$. For any $\rho < \bar{\rho}_1$, no other node has more than one link. Since g is a PNE, this implies that if $g_{ik} = 1$, either $g_{ki} = 1$ and then k is in case (b) or $\exists l \in N$ such that $g^s_{lk} = 1$ and $g_{lk} = 0$; otherwise, agent k would have incentives to cut some in-degree link off (from the definition of g^s). If $\exists l \in N$ such that $g^s_{lk} = 1$ and $g_{lk} = 0$, notice that $g_{lr} = 1$ for some $r \in N$. If $h_r < h_k$ agent l is in case (b). If $h_r > h_k$ we can repeat the same argument as before. Since n is finite, we will eventually reach an iteration in which some player would be in case (b). In consequence, if a player is in case (c), for a sufficiently low ρ there must exist a different player in case (b).

But for the link g_{ij} to be formed, node j must agree. If $o_j \leq 1$ after the deviation, player j's marginal payoff will be positive. On the other hand, let us consider the case in which $\sum_{l\in N_j}\frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ (the case in which $\frac{q+1}{\rho} - 1 \leq \sum_{l\in N_j}\frac{1}{\mu_l} < \frac{q+1}{\rho}$ is analogous). Player j's marginal payoff for forming $g_{i,j}$ is:

$$cf(h_i + h_j) > \frac{1}{\sum_{l\in N_j}\frac{1}{\mu_l}}[\sum_{l\in N_j\setminus\{j\}} c\frac{1}{\mu_l}f(h_l + h_j) + g_{jj}\frac{1}{\mu_j}f(h_j)] \tag{viii}$$

where the RHS can be interpreted as the average productivity of the ideas stored in the queue of j. Given that agent j receives some link in g^s, we can conclude that j will have a relatively high level of talent. In consequence, we can say that there should be a ρ^o such that for any $\rho < \rho^o$ the average productivity of the ideas of the queue of i depends positively on ρ (and, in consequence, it depends negatively on $\sum_{l\in N_j}\frac{1}{\mu_l}$). Therefore it should exist a ρ''' such that for any $\rho < \rho'''$ condition (viii) holds.

Given that $\rho' \leq \bar{\rho}_2$ and $\rho'' \leq \bar{\rho}_2 \leq \bar{\rho}_1$, if we define $\hat{\rho}_1$ as $min(\rho', \rho'', \rho''')$ the initial claim is proved.
■

Proof of Proposition 5. For g^s to be a PNE network, it should be robust to the following list of deviations. Next we will show that, for a sufficiently low ρ, it does.

a Let $g^s_{ij} = 1$. Let us consider that i changes the destination of her open ideas from j to herself (by definition of g^s, $h_i < h_j$). From Lemma 1, we know that if $\eta_i > 0$, then agent i will not have incentives to change her amount of in-degree links by definition of g^s. So let us consider that $\eta_i = 0$. We claim that the marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is robust to such a deviation for a sufficiently low ρ. To show it let us take the extreme (and less favorable) case in which researcher j holds $\sum_{l \in N_j} \frac{1}{\mu_l} \geq \frac{1+q}{\rho}$. In that case, i's marginal payoff will be negative when the following condition holds:

$$f(h_i) < \frac{1+q}{\rho \sum_{l \in N_j} \frac{1}{\mu_l}} cf(h_i + h_j) \tag{ix}$$

Given that j receives some link in g^s (then, she has a relatively high level of talent), we can use Corollary 1 to state that $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_j(g^s)} \frac{1}{\mu_l} < 1 + \frac{q+1}{\rho}$ for a $\rho < \bar{\rho}_2$. After some simple algebra, we can see that this is equivalent to say that $\frac{1+q}{1+q+\rho} < \frac{1+q}{\rho \sum_{l \in N_j} \frac{1}{\mu_l}} \leq \frac{1+q}{1+q-\rho}$. So, we can conclude that for $\rho < \bar{\rho}_2$, $lim_{\rho \to 0} \frac{1+q}{\rho \sum_{l \in N_j} \frac{1}{\mu_l}} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j + h_i) > f(h_i)$ for $f(\cdot)$ linear or convex, $h_i < h_j$ and $c \geq \frac{1}{2}$. Following the definition of limit, we can say that for any $\varepsilon > 0$, there always exists a $\rho' < \bar{\rho}_2$ such that $|\frac{1+q}{\rho \sum_{l \in N_j} \frac{1}{\mu_l}} - 1| < \varepsilon$, $\forall \rho < \rho'$. Then, for any h vector, and in particular for any pair (h_i, h_j) we can always find a sufficiently low value of ρ (say ρ') such that condition (ix) is hold for any $\rho < \rho'$, when $f(\cdot)$ is linear or convex. In this case, agent i will not have incentives to deviate.

b Let us consider that node i changes the destination of her open ideas from j to some other researcher with a lower talent k. This case is analogous to the previous one, thus omitted here.

c Let $g^s_{jk} = 1$. Let us consider that node j cuts the link g^s_{jk} off and proposes the link g_{ji} to agent i whose talent holds $h_i > h_k$ (the case in which agent j simply proposes an additional link to an agent i with $h_i > h_k$ is analogous). Given Lemma 1, the agent i's marginal payoff for accepting this deviation will be negative for any $\rho < \bar{\rho}_2$. Thus, g^s is robust to such a deviation for $\rho < \bar{\rho}_2$.

d A node i deletes one (or more) in-degree link. Given Lemma 1, the agent i's marginal payoff for deviating will be negative. Thus, g^s is robust to such a deviation.

e Let $g^s_{ij} = 1$. Let us consider that node i proposes the formation of the additional link g_{ik} to player k[14]. Given case (c), it only remains to analyze the case in which $h_k < h_j$. From Lemma 1, we know that if $\mu_k > 0$, for any $\rho < \bar{\rho}_2$ agent k will not have incentives to change her amount of in-degree links by definition of g^s. So, let us consider that $\mu_k = 0$. We claim that the marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is also robust to such a deviation for a sufficiently low ρ. Given that $\mu_k = 0$, player k would accept the formation of the link. But, does agent i have incentives

[14]Notice that k is not necessarily different from i.

to make such a proposal? The marginal utility obtained from that deviation would be:

$$\Delta\Pi_i = \frac{c}{\mu_i+1}[\frac{1}{\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2}}f(h_i+h_j)+\frac{\rho}{1+q}f(h_i+h_k)] - \frac{1}{\mu_i\sum_{l\in N_j}\frac{1}{\mu_l}}cf(h_i+h_j)$$

where $\mu_i = 1$. After some simple algebra, we see that $\Delta\Pi_i > 0$ if and only if:

$$f(h_i+h_k) > f(h_i+h_j)\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})} \quad \text{(x)}$$

We know that $h_k < h_j$. On the other hand, given that j has a relatively high level of talent, we can use Corollary 1 to state that $\frac{q+1}{\rho}-1 \leq \sum_{l\in N_j(g^s)}\frac{1}{\mu_l} < 1+\frac{q+1}{\rho}$ for $\rho < \bar{\rho}_2$. Then we can conclude that, for $\rho < \bar{\rho}_2$, $lim_{\rho\to o}\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})} = 1$. Since $h_k < h_j$ and following the definition of limit, we can conclude that for any vector h there always exists a sufficiently low value of ρ (say $\rho'' < \bar{\rho}_2$) such that condition (x) holds for any $\rho < \rho''$ and, as a consequence, g^s is robust to this deviation.

f Let $g^s_{ij} = 1$. Let us consider that node i deviates by forming an additional link towards herself. From Lemma 1, we know that if $\mu_i > 0$, for any $\rho < \bar{\rho}_2$ agent i will not have incentives to change her amount of in-degree links by definition of g^s. So let us consider that $\mu_i = 0$. We claim that the marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is also robust to such a deviation when ρ is sufficiently low. The marginal payoff obtained by i would be:

$$\Delta\Pi_i = \frac{1}{\mu_i+1}[\frac{c}{\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2}}f(h_i+h_j)+\frac{\rho}{1+q}f(h_i)] - \frac{c}{\mu_i}\frac{1}{\sum_{l\in N_j}\frac{1}{\mu_l}}f(h_i+h_j)$$

After some simple algebra, we see that $\Delta\Pi_i > 0$ if and only if:

$$f(h_i) > cf(h_i+h_j)\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})} \quad \text{(xi)}$$

From the definition of g^s we know that $h_i < h_j$. Repeating the same arguments as before we conclude that $lim_{\rho\to o}\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j+h_i) > f(h_i)$ for $f(\cdot)$ linear or convex, $h_i < h_j$ and $c \geq \frac{1}{2}$. Following the definition of limit, we can say that for any $\varepsilon > 0$, there always exists a ρ''' ($\rho''' < \bar{\rho}_2$) such that $|\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})}-1| < \varepsilon$, $\forall\rho < \rho'''$. Then, for any h vector, and in particular for any pair (h_i, h_j) we can always find a sufficiently low value of ρ (say ρ''') such that condition (xi) is hold for any $\rho < \rho'''$, when $f(\cdot)$ is linear or convex. In this case agent i will not have incentives to deviate; so g^s is robust to this deviation.

Defining $\hat{\rho}_2$ as $min(\rho', \rho'', \rho''')$ the claim of the proposition is proved. ■

Proof of Proposition 6. Let us divide the proof in two steps. First, we want to show that in g^s if we substitute the link g_{ij} by another one, say g_{ik}, such that $h_j > h_k$, the aggregate payoff will decrease when ρ is small enough. Given g^s, $\mu_r = 1$ and $\sum_{l \in N_r} \frac{1}{\mu_i} = \eta_r$, $\forall r \in N$. Following the definition of g^s, if $\eta_r > 0$ then η_r can take one of these two possible values, $\eta_r = \bar{\eta}$ or $\eta_r = \bar{\eta} - 1$. In the proof of lemma 1 we show that:

$$\frac{q+1}{\rho} - 1 \leq \bar{\eta} - 1 < \frac{q+1}{\rho} \leq \bar{\eta} < \frac{q+1}{\rho} + 1$$

Let us assume the less favorable case (the one in which this marginal aggregate payoff would be maximum) in which agent j has $\frac{q+1}{\rho} \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$ and $o_k < 1$ even after receiving the additional link. In such a case:

$$\triangle \sum_{i \in N} \Pi_i = \frac{\rho}{1+q}[\sum_{l \in N_j \backslash \{i,j\}} cf(h_l + h_j) + g_{jj} f(h_j)] + \frac{2\rho}{1+q} cf(h_i + h_k) -$$
$$[\frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}}[\sum_{l \in N_j \backslash \{j\}} cf(h_l + h_j) + g_{jj} f(h_j)] + \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}} cf(h_i + h_j)]$$

After some simple algebra $\triangle \sum_{i \in N} \Pi_i > 0$ if and only if:

$$(\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} - 1)[\sum_{l \in N_j \backslash \{i,j\}} cf(h_l + h_j) + g_{jj} f(h_j)] > 2[cf(h_i + h_j) - \frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} cf(h_i + h_k)]$$

Since $\frac{q+1}{\rho} \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$, $\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} \in [1, 1 + \frac{\rho}{1+q})$. Then we can say that $lim_{\rho \to 0} \frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} = 1$. Given that $h_j > h_k$, we can say that $\exists \rho'$ such that the RHS will be higher than certain ε (> 0) for any $\rho < \rho'$. On the other hand, $lim_{\rho \to 0}(\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} - 1) = 0$. In consequence, for the previous $\varepsilon > 0$, $\exists \rho''$ such that the LHS will be lower than ε for any $\rho < \rho''$. Then, we can always find a value of ρ (say ρ''') such that the last inequality will not hold for any h and for any $\rho < \rho'''$.

Second, we want to show that in g^s if we substitute the link g_{ij} by another one, say g_{ik}, such that $h_j < h_k$, the aggregate payoff will increase when ρ is small enough. Let us assume the less favorable case (the one in which this marginal aggregate payoff would be minimum) in which agent j has $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ and $o_k \geq 1$. In such a case:

$$\triangle \sum_{i \in N} \Pi_i = \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + 1}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k) + cf(h_i + h_k)) + \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + 1} cf(h_i + h_k)$$
$$+ \frac{\rho}{1+q} \sum_{l \in N_j \backslash i} cf(h_l + h_j) - \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k))$$
$$- \frac{\rho}{1+q} cf(h_i + h_j) - \frac{\rho}{1+q} \sum_{l \in N_j} cf(h_l + h_j)$$

After some simple algebra $\triangle \sum_{i \in N} \Pi_i > 0$ if and only if:

$$\frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k)) < 2(cf(h_i + h_k) - \frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} cf(h_i + h_j))$$

Since $o_k \geq 1$, we can say that $\frac{q+1}{\rho} \leq \sum_{l \in N_k} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$. Then $\frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} \in [1 + \frac{\rho}{1+q}, 1 + \frac{2\rho}{1+q})$. In consequence, $lim_{\rho \to 0} \frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} = 1$. Since $h_j < h_k$, we can say that $\exists \rho^{iv}$ such that the RHS will be higher than a certain $\varepsilon > 0$ for any $\rho < \rho^{iv}$. On the other hand, $lim_{\rho \to 0} \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}} = 0$. Then,for the previous $\varepsilon > 0$, $\exists \rho^{v}$ such that the LHS will be lower than ε for any $\rho < \rho^{v}$. Then, we can always find a value of ρ (say ρ^{vi}) such that the last inequality will hold for any h and any $\rho < \rho^{vi}$.

Defining ρ^* as $min(\rho''', \rho^{vi})$ the statement of Proposition 6 is proved. ■